TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 4

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Homework 4.1 (Absolute convergence of infinite products)**.** Let $(a_j)_{j\in\mathbf{N}}$ be a sequence in $\mathbf{C}$. We say the infinite product $\prod_{j=1}^\infty a_j$ converges absolutely if there exists $j_0 \in \mathbf{N}$ such that $a_j \neq 0$ for every $j \geq j_0$ and $(|\log a_j|)_{j\geq j_0}$ is summable, i.e.

$$\sum_{j\geq j_0} |\log a_j| < \infty.$$

a. Show absolute convergence for infinite products implies their convergence.
b. Show a product of the form $\prod_{j=1}^\infty (1+b_j)$ converges absolutely if and only if

$$\sum_{j=1}^\infty |b_j| < \infty.$$

Homework 4.2 (Examples of infinite products*)**.** Examine if the following infinite products exist in the sense of Definition 3.1. If so, calculate their value[1].

a. $\displaystyle\prod_{n=1}^\infty \Big[1 - \frac{1}{(n+1)^2}\Big].$
b. $\displaystyle\prod_{n=1}^\infty \Big[1 - \frac{1}{n}\Big]$
c. $\displaystyle\prod_{n=3}^\infty \frac{n^2-4}{n^2-1}.$
d. $\displaystyle\prod_{n=1}^\infty \frac{(1+n^{-1})^2}{1+2n^{-1}}$

Homework 4.3 (Diverging products)**.** Let $(a_j)_{j\in\mathbf{N}}$ form a sequence in $[0,+\infty)$ with the property $\sum_{j=1}^\infty (1-a_j) = \infty$. Show[2] that

$$\lim_{n\to\infty} \prod_{j=1}^n a_j = 0.$$

Homework 4.4 (A useful criterion for the convergence of infinite products)**.** Let $(a_j)_{j\in\mathbf{N}}$ be a sequence in $\mathbf{C}$. Assume $\sum_{j=1}^\infty |a_j|^2 < \infty$. Show $\prod_{j=1}^\infty (1+a_j)$ converges if and only if $\sum_{j=1}^\infty a_j$ converges. Conclude the product $\prod_{j=1}^\infty (1+z/j)$ converges if and only if $z=0$.

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[1]**Hint.** In all examples, you can directly calculate the value of the partial products.

[2]**Hint.** Use the inequality $t \leq \mathrm{e}^{t-1}$ for every $t \in \mathbf{R}$.